

# Representation of nonnegative convex polynomials

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**Abstract.** We provide a specific representation of convex polynomials non-negative on a convex (not necessarily compact) basic closed semi-algebraic set  $\mathbf{K} \subset \mathbb{R}^n$ . Namely, they belong to a specific subset of the quadratic module generated by the concave polynomials that define  $\mathbf{K}$ .

**Mathematics Subject Classification (2000).** Primary 14P10; Secondary 11E25 12D15 90C25.

**Keywords.** Positive polynomials; sums of squares; quadratic modules; convex sets.

## 1. Introduction

An important research area of real algebraic geometry is concerned with representations of polynomials positive on a basic semi-algebraic set

$$\mathbf{K} := \{x \in \mathbb{R}^n : g_j(x) \geq 0, \quad j = 1, \dots, m\} \subset \mathbb{R}^n \quad (1.1)$$

where  $g_j \in \mathbb{R}[X]$ ,  $j = 1, \dots, m$ .

An important result in this vein is Schmüdgen's Positivstellensatz [6] which states that if  $\mathbf{K}$  is compact and  $f \in \mathbb{R}[X]$  is positive on  $\mathbf{K}$  then  $f$  belongs to the preordering  $P(g)$  generated by the  $g_j$ 's; bounds on the degrees in the representation are even provided in Schweighofer [7]. Under a rather weak additional assumption on the  $g_j$ 's, Putinar's refinement [4] states that  $f$  even belongs to the quadratic module  $Q(g)$  generated by the  $g_j$ 's. The above mentioned representation results do not specialize when  $f$  is convex and the  $g_j$ 's are concave (so that  $\mathbf{K}$  is convex) a highly important case, particularly in optimization. Also, as soon as  $\mathbf{K}$  is not compact any more then negative results, notably by Scheiderer [5], exclude to represent *any*  $f$  positive on  $\mathbf{K}$  as an element of  $P(g)$  or  $Q(g)$  (except perhaps in

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This work was completed with the support of the (french) ANR grant NT05-3-41612.

low-dimensional cases). For more details, the interested reader is referred to the nice survey [5].

However, inspired and motivated by some classical results from convex optimization, we show that specialized representation results are possible when  $f$  is convex and the  $g_j$ 's are concave, in which case  $\mathbf{K} \subset \mathbb{R}^n$  is a closed (not necessarily compact) convex basic semi-algebraic set. Namely, a specific subset  $Q_c(g)$  of the quadratic module  $Q(g)$  is such that  $Q_c(g) \cap F$  is *dense* (for the  $l_1$ -norm of coefficients) in the convex cone  $F$  of convex polynomials, nonnegative on  $\mathbf{K}$ .

## 2. Convex polynomials on a convex semi-algebraic set

### 2.1. Notation and Preliminaries

Let  $\mathbb{R}[X]$  be the ring of real polynomials in the variables  $X = (X_1, \dots, X_n)$ , and let  $\Sigma^2 \subset \mathbb{R}[X]$  be the subset of sums of squares (sos) polynomials. If  $f \in \mathbb{R}[X]$ , write  $f(X) = \sum_{\alpha \in \mathbb{N}^n} f_\alpha X^\alpha$ , and denote its  $l_1$ -norm by  $\|f\|_1 (= \sum_{\alpha \in \mathbb{N}^n} |f_\alpha|)$ .

Let  $Q(g) \subset \mathbb{R}[X]$  be the *quadratic module* generated by a set of polynomials  $g = (g_j)_{j=1}^m \subset \mathbb{R}[X]$ , that is,

$$Q(g) := \left\{ \sigma_0 + \sum_{j=1}^m \sigma_j g_j : \sigma_j \in \Sigma^2, j = 0, \dots, m \right\}. \quad (2.1)$$

Throughout the paper we make the following assumption.

**Assumption 2.1.**  $\mathbf{K} \subset \mathbb{R}^n$  is defined in (1.1) and is such that:

- (a)  $g_j$  is concave for every  $j = 1, \dots, m$ .
- (b) There exists  $z \in \mathbf{K}$  such that  $g_j(z) > 0$  for every  $j = 1, \dots, m$ .

Assumption 2.1(b), known as Slater condition, is an important regularity condition for the celebrated Karush-Kuhn-Tucker optimality conditions.

**Proposition 2.2.** Let Assumption 2.1 hold and let  $f \in \mathbb{R}[X]$  be convex and such that  $f^* := \inf_x \{f(x) : x \in \mathbf{K}\} = f(x^*)$  for some  $x^* \in \mathbf{K}$ .

Then there exists  $\lambda \in \mathbb{R}_+^m$  such that

$$\nabla f(x^*) - \sum_{j=1}^m \lambda_j \nabla g_j(x^*) = 0; \quad \lambda_j g_j(x^*) = 0, \quad j = 1, \dots, m. \quad (2.2)$$

In other words, the Lagrangian  $L_f \in \mathbb{R}[X]$  defined by

$$X \mapsto L_f(X) := f(X) - f^* - \sum_{j=1}^m \lambda_j g_j(X), \quad X \in \mathbb{R}^n, \quad (2.3)$$

is a nonnegative polynomial which satisfies

$$L_f(x^*) = 0; \quad \nabla L_f(x^*) = 0. \quad (2.4)$$

See e.g. Polyak [3].

## 2.2. Convex Positivstellensatz

If one is interested in representation of polynomials nonnegative on  $\mathbf{K}$ , the first polynomial to consider is of course  $f - f^*$  where  $0 \leq f^* = \inf_{x \in \mathbf{K}} f(x)$ . Indeed, any other positive polynomial is just  $(f - f^*) + f^*$  with  $f^* \geq 0$ . And so, if  $f - f^*$  belongs to some preordering or some quadratic module, then so does  $f$ . From Proposition 2.2 it is easy to establish the following result.

**Corollary 2.3.** *Let Assumption 2.1 hold and let  $f \in \mathbb{R}[X]$  be convex and such that  $f^* := \inf_x \{f(x) : x \in \mathbf{K}\} = f(x^*)$  for some  $x^* \in \mathbf{K}$ . If the nonnegative polynomial  $L_f$  of (2.3) is sos then*

$$f - f^* = \sigma + \sum_{j=1}^n \lambda_j g_j \quad (2.5)$$

for some convex sos polynomial  $\sigma \in \Sigma^2$  and some nonnegative scalars  $\lambda_j$ ,  $j = 1, \dots, m$ . That is,  $f - f^* \in Q(g)$ , with  $Q(g)$  as in (2.1). In addition, the sos weights associated with the  $g_j$ 's are just nonnegative constants, and  $\sigma$  is convex.

*Proof.* Follows from the definition (2.3) of  $L_f$ , and the fact that  $L_f$  is sos.  $\square$

Hence in view of Corollary 2.3, an interesting issue is to provide sufficient conditions for  $L_f$  to be sos. For instance, consider the following definition from Helton and Nie [1]

**Definition 2.4 (Helton and Nie [1]).** A polynomial  $f \in \mathbb{R}[X]$  is sos-convex if its Hessian  $\nabla^2 f$  is a sum of squares (sos), that is, there is some integer  $p$  and some matrix polynomial  $F \in \mathbb{R}[X]^{p \times n}$  such that

$$\nabla^2 f(X) := \left( \frac{\partial^2 f(X)}{\partial X_i \partial X_j} \right)_{ij} = F(X)^T F(X). \quad (2.6)$$

**Corollary 2.5.** *Let Assumption 2.1 hold, and let  $f \in \mathbb{R}[X]$  be convex and such that  $f^* := \inf_x \{f(x) : x \in \mathbf{K}\} = f(x^*)$  for some  $x^* \in \mathbf{K}$ .*

*If  $f$  is sos-convex and  $-g_j$  is sos-convex for every  $j = 1, \dots, m$ , then  $f - f^* \in Q(g)$ . More precisely, (2.5) holds for some convex sos polynomial  $\sigma \in \Sigma^2$  and some nonnegative scalars  $\lambda_j$ ,  $j = 1, \dots, m$ .*

*Proof.* From Proposition 2.2, let  $L_f$  be as in (2.3). As  $f$  and  $-g_j$  are sos convex, write

$$\nabla^2 f(X) = F(X)^T F(X); \quad -\nabla^2 g_j(X) = G_j(X)^T G_j(X), \quad j = 1, \dots, m,$$

for some  $F \in \mathbb{R}[X]^{p \times n}$  and some  $G_j \in \mathbb{R}[X]^{p_j \times n}$ ,  $j = 1, \dots, m$ . Hence,

$$\nabla^2 L_f = \nabla^2 f - \sum_{j=1}^m \lambda_j \nabla^2 g_j = F^T F + \sum_{j=1}^m \lambda_j G_j^T G_j = H^T H,$$

with  $H^T := [F^T \mid \sqrt{\lambda_1} G_1^T \mid \dots \mid \sqrt{\lambda_m} G_m^T]$ , and so  $L_f$  is sos-convex. As (2.4) holds, by Lemma 3.2 in Helton and Nie [1], the polynomial  $L_f$  is sos, and so, by Corollary 2.3, the desired result (2.5) holds.  $\square$

Next, consider the subset  $Q_c(g) \subset Q(g)$  defined by:

$$Q_c(g) := \left\{ \sigma + \sum_{j=1}^m \lambda_j g_j : \lambda \in \mathbb{R}_+^m; \sigma \in \Sigma^2, \sigma \text{ convex.} \right\} \subset Q(g). \quad (2.7)$$

The set  $Q_c(g)$  is a specialization of  $Q(g)$  to the convex case, in that the weights associated with the  $g_j$ 's are nonnegative scalars, i.e., sos polynomials of degree 0, and the sos polynomial  $\sigma$  is convex.

**Theorem 2.6.** *Let Assumption 2.1 hold, and let  $Q_c(g)$  be as in (2.7). Let  $F \subset \mathbb{R}[X]$  be the convex cone of convex polynomials nonnegative on  $\mathbf{K}$ .*

*Then  $Q_c(g) \cap F$  is dense in  $F$  for the  $l_1$ -norm  $\|\cdot\|_1$ . In particular, if  $\mathbf{K} = \mathbb{R}^n$  (so that  $F$  is now the set of nonnegative convex polynomials), then  $\Sigma^2 \cap F$  is dense in  $F$ .*

*Proof.* Let  $f \in F$  and let  $r_0 := \lfloor (\deg f)/2 \rfloor + 1$ . Given  $r \in \mathbb{N}$ , let  $\Theta_r \in \mathbb{R}[X]$  be the polynomial

$$X \mapsto \Theta_r(X) := 1 + \sum_{i=1}^n X_i^{2r}. \quad (2.8)$$

For every  $\epsilon > 0$ , the polynomial  $f_{\epsilon 0}(X) := f(X) + \epsilon \Theta_{r_0}(X)$  is convex and nonnegative on  $\mathbf{K}$ , i.e.,  $f_{\epsilon 0} \in F$ . In addition,

$$0 \leq f^* := \inf_{x \in \mathbf{K}} f(x) \leq \inf_{x \in \mathbf{K}} f_{\epsilon 0}(x) = f_{\epsilon 0}(x_\epsilon^*) =: f_\epsilon^*,$$

for some  $x_\epsilon^* \in \mathbf{K}$ . Indeed, the level set  $\{x \in \mathbf{K} : f_{\epsilon 0}(x) \leq \alpha\}$  is compact for every  $\alpha \in \mathbb{R}$ , and so,  $f_{\epsilon 0}$  attains its minimum on  $\mathbf{K}$ . Obviously, we also have  $\|f_{\epsilon 0} - f\|_1 \rightarrow 0$  as  $\epsilon \downarrow 0$ . Next, let  $L_{f_{\epsilon 0}}$  be as in (2.3), i.e.,

$$L_{f_{\epsilon 0}} = f + \epsilon \Theta_{r_0} - f_\epsilon^* - \sum_{j=1}^n \lambda_j^\epsilon g_j,$$

for some nonnegative vector  $\lambda_j^\epsilon \in \mathbb{R}_+^m$ . As  $L_{f_{\epsilon 0}} \geq 0$  on  $\mathbb{R}^n$ , by Corollary 3.3 in Lasserre and Netzer [2], there exists  $r_\epsilon \in \mathbb{N}$  such that for every  $r \geq r_\epsilon$ ,  $L_{f_{\epsilon 0}} + \epsilon \Theta_r$  is sos. That is,  $\sigma := L_{f_{\epsilon 0}} + \epsilon \Theta_r \in \Sigma^2$  and so

$$f_\epsilon := f + \epsilon (\Theta_{r_0} + \Theta_r) = \sigma + f_\epsilon^* + \sum_{j=1}^n \lambda_j^\epsilon g_j.$$

Notice that by definition,  $\sigma \in \Sigma^2$  is convex. Next, as  $f_\epsilon^* \geq 0$ ,  $\sigma + f_\epsilon^* \in \Sigma^2$ , and so, equivalently,  $f_\epsilon \in Q_c(g)$ .

In addition,  $f_\epsilon \in F$  because  $f_\epsilon$  is convex (as  $f_\epsilon = f + \epsilon(\Theta_{r_0} + \Theta_r)$ ) and nonnegative on  $\mathbf{K}$  (as  $f_\epsilon \geq f$ ), and so,  $f_\epsilon \in Q_c(g) \cap F$ . Finally,  $\|f - f_\epsilon\|_1 = \epsilon \|\Theta_{r_0} + \Theta_r\|_1 \rightarrow 0$  as  $\epsilon \downarrow 0$ .

Finally, if  $\mathbf{K} = \mathbb{R}^n$  (so that  $F$  is now the set of nonnegative convex polynomials), one obtains  $Q_c(g) = \Sigma^2$ .  $\square$

One may also replace  $\Theta_r$  in (2.8) with the new perturbation

$$X \mapsto \theta_r(X) := \sum_{k=0}^r \sum_{j=1}^n \frac{X_j^{2k}}{k!}.$$

This perturbation also preserves convexity. In addition, not only  $\|f - f_\epsilon\|_1 \rightarrow 0$  as  $\epsilon \downarrow 0$ , but the convergence  $f_\epsilon \rightarrow f$  is also *uniform* on compact sets!

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